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A rigorous real-time Feynman path integral and propagator

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Abstract. We will derive a rigorous real-time propagator for the non-relativistic quantum mechanical L^2 transition probability amplitude and for the non-relativistic wavefunction. The propagator will be given explicitly in terms of the time evolution operator. The derivation will be for all self-adjoint non-vector potential Hamiltonians. For systems with potentials that carry at most a finite number of singularity and discontinuities, we will show that our propagator can be written in the form of a rigorous real-time, time-sliced Feynman path integral via improper Riemann integrals. We will also derive the Feynman path integral in a non-standard analysis formulation. Finally, we will compute the propagator for the harmonic oscillator using the non-standard analysis Feynman path-integral formulation; we will compute the propagator without using any knowledge of the classical properties of the harmonic oscillator.

1. Introduction

Since Feynman's invention of the path integral, much research have been done to make the real-time Feynman path integral mathematically rigorous (see [6, 9, 10, 13, 18–20]). In physics, the real-time, time-sliced Feynman path integral is formally given by (see [3–5])

$$\begin{aligned}
&\bar{K}_t(\vec{x},\vec{x}_0) = \lim_{k\to\infty} w_{n,k}\int_{r\mathbb{R}^{(k-1)n}} \exp\left[\frac{\mathrm{i}\epsilon}{\hbar}S_k(\vec{x}=\vec{x}_k,\ldots,\vec{x}_o)\right]\mathrm{d}\vec{x}_1\ldots\mathrm{d}\vec{x}_{k-1}\\
&\vec{x}_0=\vec{q}_0 \qquad \vec{x}_{k+1}=\vec{q} \qquad \epsilon=\frac{t}{k} \qquad w_{n,k}=\left(\frac{m}{2\mathrm{i}\pi\hbar\epsilon}\right)^{n(k+1)/2}\\
&S\{\vec{x}_{k+1}\ldots\vec{x}_0\} = \sum_{j=1}^{k+1}\left[\frac{m}{2}\left(\frac{\vec{x}_j-\vec{x}_{j-1}}{\epsilon}\right)^2 - V(\vec{x}_j)\right]\\
&\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \int \bar{K}_t(\vec{x},\vec{x}_0)\,\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\\
&\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\delta(\vec{y}-\vec{x}_0)\right](\vec{x}) = \bar{K}_t(\vec{x},\vec{x}_0)
\end{aligned} \tag{1.1}$$

where the integral of the first equation in (1.1) is an improper Riemann integral, and the last line in (1.1) is the evolution operator operating on the dirac delta function's $\vec{y}$ variable. It is well known that mathematical rigour of (1.1) is lacking, and we know that an integral over path space in real time cannot be well defined with measure theory (see [6]).

The problems with the objects in equation (1.1) are that we do not know whether the improper Riemann integrals exist, we do not know whether the k limit exists, and we do not know whether the Feynman path integral in (1.1) produces the propagator. In his paper (see [11], footnote 13), Feynman observed that by using wavefunctions, ill-defined oscillatory integrals can be given a rigorous meaning. With this observation, we will reformulate equation (1.1) into rigorous mathematical objects that represent the propagator.

From mathematics, we know that for some values of t, some propagators must be treated as distributions; the harmonic oscillator is one such example (see [6, 7]). Also, $\bar{K}_t\,(\vec{x}, \vec{x}_0)$ given in (1.1) is a function of $\vec{x}$ and $\vec{x}_0$. Thus, it is natural to consider $\bar{K}_t$ as a tempered distribution on the class of Schwartz test functions $S\,(\mathbb{R}^n \times \mathbb{R}^n)$. The space of square-integrable functions is a subset of the space of tempered distributions. If we consider the wavefunction as a distribution and take its inner produce with a test function, we can formally use (1.1) and obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x} &= \int_{\mathbb{R}^n} \phi(\vec{x}) \int \bar{K}_t\,(\vec{x}, \vec{x}_0)\,\psi\big(\vec{x}_0\big)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x} \\ &= \int \bar{K}_t\,(\vec{x}, \vec{x}_0)\,\phi(\vec{x})\psi\big(\vec{x}_0\big)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0. \end{aligned} \tag{1.2}$$

Equation (1.2) will form theorem 2.5.

As for the formal evolution of the delta function in (1.1), let us consider formally the following equation:

$$\begin{aligned}\lim_{\eta,\gamma\to 0} K(\vec{x}, \vec{x}_0, \eta, \gamma, t) &= \lim_{\eta,\gamma\to 0} \int_{\mathbb{R}^n} G_x(\vec{y}, \eta) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) F_{\vec{x}_0}(\vec{z}, \gamma)\right](\vec{y})\,\mathrm{d}\vec{y} \\ &= \int \delta\,(\vec{x} - \vec{y}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\delta\,(\vec{z} - \vec{x}_0)\right](\vec{y})\,\mathrm{d}\vec{y} \\ &= \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\delta\,(\vec{z} - \vec{x}_0)\right](\vec{x}) \end{aligned} \tag{1.3}$$

where the functions K, F and G are given by equation (2.1). If we are going to take the propagator as a distribution in the sense of (1.2), we might consider the limit in (1.3) as a distribution limit. Doing so produces theorem 2.2 and in some sense theorem 2.4 (equations (2.2), (2.3), (2.6*a*) and (2.6*b*)).

In mathematics, there exists a rigorous formulation for a real-time, time-sliced Feynman path integral (see [7, 8]), it reads

$$\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \lim_{k\to\infty} w_{n,k} \int_{\mathbb{R}^{kn}} \exp\left[\frac{\mathrm{i}\epsilon}{\hbar} S_k\,(\vec{x}_k = \vec{x}, \ldots, \vec{x}_0)\right]\psi\big(\vec{x}_0\big)\,\mathrm{d}\vec{x}_0 \ldots \mathrm{d}\vec{x}_{k-1} \tag{1.4}$$

where $\psi \in L^2$, the integral in (1.4) is an improper Lebesgue integral with convergence taken in the L^2 topology, and the k limit in (1.4) is taken in the L^2 topology. Comparing (1.4) and (1.1), we see that rigorously we have a Feynman path integral that has all convergence taken in L^2 topology while in physics, a formally improper Riemann integral and pointwise convergence are favoured.

What we will do is convert all convergences in L^2 topology into pointwise convergences in t. The idea is to use a wavefunction as a convergence factor as observed by Feynman. For simplicity, suppose $f(x) \in L^2(\mathbb{R})$, $g(x, y) \in L^2\,(\mathbb{R} \times \mathbb{R})$ are such that they are bounded and continuous. Furthermore, suppose that both

$$\begin{aligned} h(x) &= \int_{-a}^{b} g(x, y)\,\mathrm{d}y \\ p(x) &= \lim_{a,b\to\infty} \int_{-a}^{b} g(x, y)\,\mathrm{d}y \end{aligned} \tag{1.5}$$

are in $L^2(\mathbb{R})$ as a function of x. In (1.5), we take the integrals to be Lebesgue integrals and the limits are taken independently of each other in the L^2 norm. Note that for $p(x)$, we can interpret the integral as an improper Lebesgue integral with convergence in the L^2 topology. Let us denote by $\chi_{[-c,d]}$ the characteristic function on $[-c, d]$. Schwarz's inequality then implies

$$\left| \int_{\mathbb{R}} f(x)p(x)\,\mathrm{d}x - \int_{-c}^{d}\int_{-a}^{b} f(x)g(x,y)\,\mathrm{d}x\,\mathrm{d}y \right|$$
$$\leqslant \|f\|_2\|p-h\|_2 + \|f - \chi_{[-c,d]}f\|_2\|h\|_2 \to 0. \tag{1.6}$$

Thus, we can write

$$\int_{\mathbb{R}} f(x)p(x)\,\mathrm{d}x = \lim_{a,b,c,d\to\infty} \int_{-c}^{d}\int_{-a}^{b} f(x)g(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{1.7}$$

where the limits are all taken independently of each other. Since f and g are bounded and continuous, the Lebesgue integral over $[-a, b]\times[-c, d]$ in (1.7) can be replaced by a Riemann integral. Since the limits are taken independently of each other, we can then interpret the right hand-side of (1.7) as an improper Riemann integral. If f and g carry singularities and discontinuities, care must be taken in the region of integration so that the replacement of the Lebesgue integral by Riemann integrals can be done.

The technique of converting L^2 limits into pointwise limits as illustrated above is what we will use to prove all of the theorems in the next section. It is the foundation of this work.

2. Results

In his paper (see [11], footnote 13), Feynman observed that by using wavefunctions, ill-defined oscillatory integrals can be given a rigorous meaning. With this observation, we will reformulate equation (1.1) into a rigorous mathematical object that represents the propagator.

The goal of this paper is the following. First, we will elaborate on Feynman's observation and use wavefunctions to provide a convergence factor in the derivation of a real-time propagator that takes the form of an L^2 transition probability amplitude. We will use wavefunctions to derive a real-time, time-sliced Feynman path integral. We will derive two non-standard analysis formulations of the time-sliced Feynman path integral. Finally, we will compute the propagator of the harmonic oscillator using our non-standard Feynman path-integral representation. We will assume that the reader is familiar with non-standard analysis (see [13–17] and references therein).

The usual idea in using non-standard analysis is to replace the time-slice limit by a standard part (see [9, 13, 18, 19] and references therein). We will derive a non-standard formulation that transfers the time-slice limit into the non-standard world and standard part is taken on infinitesimal parameters in wavefunctions. It was shown in [19] that for the harmonic oscillator, equation (1.1) can be cast in the language of non-standard analysis where the time-slice limit is replaced by a standard part. Furthermore, using non-standard analysis methods, one can rigorously compute the harmonic oscillator propagator without having prior knowledge of the classical path. We will follow the approach of [19] in the computations of this paper. In [19], we do not know whether equation (1.1) is the propagator *a priori*; we are satisfied because the computation produced the correct results. In this paper, we will have a Feynman path-integral representation that is known to produce the propagator and we will use it to compute the harmonic oscillator propagator.

What we will show is the following. Let $H = \frac{-\hbar^2}{2m}\Delta + V(\vec{x}) = H_0 + V(\vec{x})$ be essentially self-adjoint and the domain of H contains the Schwartz space of rapidly decreasing test functions. Denote the closure of H by $\bar{H}$. Let $t > 0$ and let

$$\begin{aligned} F_{\vec{x}}(\vec{y},\gamma) &= F(\vec{x},\vec{y},\gamma) = \left(\frac{m}{2\pi\hbar\gamma}\right)^{n/2} \exp\left[\frac{-m\gamma}{2\hbar}\left(\frac{\vec{x}-\vec{y}}{\gamma}\right)^2\right] \qquad \gamma > 0 \\ G_{\vec{x}}(\vec{y},\eta) &= G(\vec{x},\vec{y},\eta) = \left(\frac{m}{2\pi\hbar\eta}\right)^{n/2} \exp\left[\frac{-m\eta}{2\hbar}\left(\frac{\vec{x}-\vec{y}}{\eta}\right)^2\right] \qquad \eta > 0 \\ K(\vec{x},\vec{x}_0,\eta,\gamma,t) &= \int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y},\eta)\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)F_{\vec{x}_0}(\vec{z},\gamma)\right](\vec{y})\,\mathrm{d}\vec{y}. \end{aligned} \tag{2.1}$$

The notation for $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is that the evolution operator operates on the $\vec{z}$ variable while leaving $\vec{x}_0$ fixed and the result is a function of $\vec{x}_0$ and $\vec{y}$; finally, the $\vec{y}$ variable is integrated over G. We point out that K in (2.1) is in the form of an L^2 transition probability amplitude, but neither F nor G are wavefunctions since they are not normalized to 1 in the L^2 norm. Also, the form of K is similar to the form of the propagator given in [12] where Prugovecki provides a theory of stochastic quantum mechanics. It will be interesting to see the relationship between (2.1) and the stochastic propagator derived by Prugovecki. One immediate difference between (2.1) and Progovecki's formulation is that (2.1) stays within the popular representation of quantum mechanics, whereas Prugovecki uses a different representation (see [12]).

The existence of $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is immediately obvious since both functions in the integrand are in L^2. We will show that

Theorem 2.1. *$K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ is continuous as a function of $(\vec{x},\vec{x}_0) \in \mathbb{R}^{2n}$ and it is uniformly bounded as a function of $(\vec{x},\vec{x}_0) \in \mathbb{R}^{2n}$.*

The kernel in theorem 2.1 will play the role of an integral kernel in the following sense.

Theorem 2.2(a). *Let $\phi, \psi \in L^2(\mathbb{R}^n)$ Let H be essentially self-adjoint, then*

$$\int_{\mathbb{R}^n} \phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x} = \lim_{\eta,\gamma\to 0} \bar{\int}_{\mathbb{R}^{2n}} \phi(\vec{x})\psi(\vec{x}_0)K(\vec{x},\vec{x}_0,\eta,\gamma,t)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x} \tag{2.2}$$

where the notation for the integral on the right-hand side of the equality in (2.2) means an improper Lebesgue integral with the convergence at infinity taken pointwise in t (see equation (5.10) for more details), and the limits are taken independently of each other and pointwise in t.

Note that in the above theorem, the convergence of the improper Lebesgue integral is pointwise in t as opposed to convergence in the L^2 topology in equation (1.4). A pointwise convergence might provide computational advantages.

We will not attempt to pass the limits in (2.2) inside the integral since some real-time propagators do not exist as a function for all time and must be treated as distributions (see [1]). We will make a connection between $K(\vec{x},\vec{x}_0,\eta,\gamma,t)$ and the theory of distributions (see remark 6.2 and equation (6.4)). On the other hand, we would be like to be able to pass the limits inside the improper Lebesgue integral when the kernel in theorem 2.1 and the propagator for the evolution are well behaved. The next theorem provides us with that opportunity.

Theorem 2.2(b). *Let $\phi, \psi \in L^2 \bigcap L^1$. Let H be essentially self-adjoint, then*

$$\int_{\mathbb{R}^n} \phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x} = \lim_{\eta,\gamma\to 0} \int_{\mathbb{R}^{2n}} \phi(\vec{x})\psi(\vec{x}_0)K(\vec{x},\vec{x}_0,\eta,\gamma,t)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x} \tag{2.3}$$

where the integral on the right-hand side of the equality is a Lebesgue integral and all limits are taken independently of each other and pointwise in t.

In the above theorem, theorem 2.1 and the fact that the wavefunctions are in L^1 provide us with the opportunity to pass the limits inside the integral and produce the propagator for the evolution. We will do this for the harmonic oscillator Hamiltonian. Furthermore, for the purpose of passing the limits, we will not attempt to generalize the wavefunctions to all of L^2 since integrating the propagator over two arbitrary L^2 wavefunctions in the sense of equation (1.2) is not always well defined; the free evolution propagator is one such example.

The kernel in (2.1) can be represented explicitly by a time-sliced Feynman path integral.

Theorem 2.3. *Let H be essentially self-adjoint, and the potential V be such that it has at most a finite number of discontinuities and singularities. Let*

$$w_{n,k} = \left(\frac{m}{2\mathrm{i}\pi\hbar\epsilon}\right)^{nk/2} \qquad \epsilon = \frac{t}{k}$$
$$S_k\left(\vec{x}_{k+1}, \ldots, \vec{x}_1\right) = \sum_{j=2}^{k+1}\left[\frac{m}{2}\left(\frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon}\right)^2 - V\left(\vec{x}_j\right)\right] \tag{2.4}$$

then

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = \lim_{k\to\infty} w_{n,k}\int_{r\mathbb{R}^{(k+1)n}} F_{\vec{x}_0}(\vec{x}_1, \gamma) \times \exp\left[\frac{\mathrm{i}\epsilon}{\hbar}S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1)\right] G_{\vec{x}}(\vec{x}_{k+1}, \eta)\,\mathrm{d}\vec{x}_1 \ldots \mathrm{d}\vec{x}_{k+1}. \tag{2.5}$$

In (2.5), the integral is an improper Riemann integral and the k limit is taken pointwise in t.

In theorem 2.3, there is no restriction on the type of discontinuities and singularities on the potential as long as the Hamiltonian is essentially self-adjoint, and by an improper Riemann integral we mean a Riemann integral with convergence at infinity taken pointwise in t. Furthermore, it is not necessary to put the restriction on the potential in the above theorem and work with an improper Riemann integrals if one is willing to work with improper Lebesgue integral as in theorem 2.2(a) (see remark 3.2), but for our purpose of computing the harmonic oscillator propagator (see section 8), we will formulate theorem 2.3 as above.

The problem with formulating a real-time, time-sliced Feynman path integral is that Fubini's theorem cannot be applied due to the oscillatory nature of the integrand. We will see that the application of Fubini's theorem can be justified in the derivation of (2.5) because functions F and G play the role of convergence factors as Feynman pointed out.

The propagator is usually formulated for the wavefunction. If we wish to work with the wavefunction, we have the following.

Theorem 2.4(a). *Let $\psi \in L^2(\mathbb{R}^n)$, H be essentially self-adjoint, then the following is true:*

$$\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^n}\psi\left(\vec{x}_0\right)K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\,\mathrm{d}\vec{x}_0 \tag{2.6a}$$

where the integral in (2.6a) is a Lebesgue integral and the limits are taken independently of each other in the L^2 topology.

Theorem 2.4(a) above provides us with another way to deal with arbitrary L^2 wavefunctions when equation (2.1) is considered as a distribution in $\mathbb{R}^{2n}$.

Theorem 2.4(b). *Let $\psi, \phi \in L^2(\mathbb{R}^n)$, H be essentially self-adjoint, then the following is true:*

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \lim_{\eta,\gamma\to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left(\int_{\mathbb{R}^n} \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\right) \mathrm{d}\vec{x} \tag{2.6b}$$

where the integrals in (2.6b) are iterated Lebesgue integrals and the limits are taken independently of each other and pointwise in t.

We will connect theorem 2.2 to the theory of distributions.

Theorem 2.5. *Let S be the space of rapidly decreasing test functions. Suppose $H = H_0 + V$ is essentially self-adjoint, then there exists a tempered distribution $K_t(\vec{x}, \vec{x}_0)$ on $S(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\forall \phi(\vec{x}), \psi(\vec{x}_0) \in S(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\, \mathrm{d}\vec{x} = \int K_t(\vec{x}, \vec{x}_0)\, \phi(\vec{x})\psi(\vec{x}_0)\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{x}_0 \tag{2.7}$$

where the integral on the right-hand side of equation (2.7) is a distribution inner product.

Theorem 2.2 above is linked to the theory of tempered distributions via theorem 2.5. We will prove some properties of the distribution $K_t(\vec{x}, \vec{x}_0)$ given in (2.7). We have, in fact, gone beyond the theory of distributions in the sense that theorems 2.2 and 2.4 are not just true for rapidly decreasing test functions, they are true for a much bigger class of functions.

Finally, the above theorems can be cast into the language of non-standard analysis. The idea of using non-standard analysis to formulate the Feynman path integral is not new (see [9, 13, 18, 19] and references therein). The usual formulation is to replace the time-slice limit with a standard part. Following that idea, theorem 2.3 can be easily reformulated in the language of non-standard analysis as follows.

Theorem 2.6. *With the notation and conditions in theorem 2.3, we can write*

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = \lim_{k\to\infty} K_k(\vec{x}, \vec{x}_0, \eta, \gamma, t) \tag{2.8}$$

where K_k is given in (2.5). Let $\omega \in {}^*\mathbb{N} - \mathbb{N}$, then

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = st({}^*K_\omega(\vec{x}, \vec{x}_0, \eta, \gamma, t)) \tag{2.9}$$

where st is the standard part.

Note that ${}^*K_\omega$ consists of an infinite (namely ω) copies of *-improper Riemann integrals. Equation (2.9) is, in fact, a special case of the work done in [9].

We now come to a new formulation of the Feynman path integral in the language of non-standard analysis.

Theorem 2.7. *Under the conditions of theorem 2.2 (part (a) or (b)), let η and γ be positive infinitesimal in the language of non-standard analysis, then*

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\, \mathrm{d}\vec{x} = st\left({}^*\bar{\int}_{\mathbb{R}^{2n}} \phi(\vec{x})\psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\, \mathrm{d}\vec{x}\right) \tag{2.10a}$$

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\, \mathrm{d}\vec{x} = st\left({}^*\int_{\mathbb{R}^{2n}} \phi(\vec{x})\psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\, \mathrm{d}\vec{x}\right) \tag{2.10b}$$

*where equations (2.10a) and (2.10b) correspond to theorem 2.2(a) and (b), respectively, st denotes the standard part of the *-transformed improper Lebesgue integral ((2.10a), in the sense of theorem 2.2(a)) and the Lebesgue integral ((2.10b), theorem 2.2(b)).*

The implication of theorem 2.7 on Feynman path integrals is the following. Suppose the Hamiltonian has a finite number of singularities and discontinuities, then theorem 2.3 holds. *-transforming theorem 2.3, equation (2.5) reads: for all $\eta, \gamma \in {}^*\mathbb{R}^+$,

$$^*K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = {}^*\lim_{k\to\infty} {}^*K_k(\vec{x}, \vec{x}_0, \eta, \gamma, t) \tag{2.11}$$

where the *-limit is a limit taken in the non-standard world, and $^*K_k$ are k copies of *-improper Riemann integrals. In particular, we can let η and γ be positive infinitesimal and use equation (2.11) in theorem 2.7. The time-sliced Feynman path integral now has time-sliced limits taken in the non-standard world and standard parts taken on η and γ in the sense of theorem 2.7. As mentioned earlier, this formulation differs from the popular usage of non-standard analysis on path integrals in that the time-slice limit is not replaced by a standard part. This new formulation uses the fact that functions F and G behave like delta functions when η and γ are positive infinitesimal.

Lastly, we will use equation (2.9) and theorem 2.2(b) to compute the harmonic oscillator propagator. We will compute the propagator in such a way that no prior knowledge of the classical path is needed. In fact, the classical part of the propagator naturally falls out from quantum considerations. The usual method to compute the harmonic oscillator with the Feynman path integral is to use the classical path and separate out the classical and quantum fluctuation parts (see [4, 5]). From our computational point of view, the classical mechanics part comes purely from quantum considerations and that goes against the grain of Feynman's original idea that quantum mechanics comes from classical mechanics via the action integral in the integrand of integration over path space.

The Hamiltonian for the harmonic oscillator is

$$H = \frac{-\hbar^2}{2m}\Delta + \frac{m\lambda^2}{2}\vec{x}^2.$$

It is well known that for $0 < t < \pi/\lambda$, the n-dimensional harmonic oscillator propagator is given by

$$\begin{aligned}
K(\vec{x}, \vec{x}_0, t) &= \left(\frac{m}{2\pi i\hbar}\right)^{n/2}\left(\frac{\lambda}{\sin\lambda t}\right)^{n/2}\exp\left\{\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^2 + \vec{x}^2\right)\cos\lambda t - 2\vec{x}\vec{x}_0\right]\right\}\\
&= h\left(\tfrac{1}{2}n, t\right) g\left(\vec{x}, \vec{x}_0, t\right)\\
h\left(\tfrac{1}{2}n, t\right) &= \left(\frac{m}{2\pi i\hbar}\right)^{n/2}\left(\frac{\lambda}{\sin\lambda t}\right)^{n/2}\\
g\left(\vec{x}, \vec{x}_0, t\right) &= \exp\left\{\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^2 + \vec{x}^2\right)\cos\lambda t - 2\vec{x}\vec{x}_0\right]\right\}
\end{aligned} \tag{2.12}$$

where $g(\vec{x}, \vec{x}_0, t)$ is the classical part and $h(n/2)$ is the quantum fluctuation. Given (2.12), we would expect that

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = h\left(\tfrac{1}{2}n, t\right)\int_{\mathbb{R}^{2n}} g(\vec{y}, \vec{y}_0, t)F(\vec{y}_0, \vec{x}_0, \gamma)G(\vec{y}, \vec{x}, \eta)\,\mathrm{d}\vec{y}\,\mathrm{d}\vec{y}_0. \tag{2.13}$$

Note that in (2.13), the disturbance of the functions F and G affects only the classical part.

We conclude this section with a comment and a summary. The reader should compare the similarities and differences between the formulations above and that of the notion of weak integral kernels (see [2] and references therein). In particular, one difference is that the above kernel exists for all essentially self-adjoint Hamiltonians. The main purpose of this paper is to derive a rigorous theory of real-time propagators and real-time Feynman path integrals. The propagator exists for all essentially self-adjoint Hamiltonians and is closely related to distributions. The Feynman path integral exists for potentials that carry at most a finite number of singularities and discontinuities, it is formulated via improper Riemann integrals, and it can be formulated with classical analysis and non-standard analysis. Lastly, we use non-standard analysis and compute the propagator for the harmonic oscillator without prior knowledge of the classical path.

3. Proof of theorem 2.3

We start by giving a quick proof of theorem 2.3. This theorem is a specific case of the work done in [9]. For full details of the proof, we refer the reader to [9].

We first set some notation. Suppose V is such that it has at most a finite number of singularities and discontinuities. Let $k \in \mathbb{N}$ and $1 \leqslant l \leqslant k+1$. We will denote the interior of the lth box by

$$A^l = (-a_1^l, b_1^l) \times \cdots \times (-a_n^l, b_n^l) \tag{3.1}$$

for positive and large a and b. Let $K = \{\vec{y}_1 \ldots \vec{y}_p\}$ be the set of discontinuous and singular points of V. For each $\vec{y}_q = (y_1^q, \ldots, y_n^q) \in K$, denote the lth box centred at $\vec{y}_q$ by

$$B_q^l = \left(y_1^q - \frac{1}{c_1^{q,l}}, y_1^q + \frac{1}{d_1^{q,l}}\right) \times \cdots \times \left(y_n^q - \frac{1}{c_n^{q,l}}, y_n^q + \frac{1}{d_n^{q,l}}\right) \tag{3.2}$$

for positive and large c and d. Let

$$C^l = A^l - \left\{\bigcup_{q=1}^{p} B_q^l\right\}. \tag{3.3}$$

For arbitrary large a, b, c and d, C^l is a box which encloses the set K and at each point of K, a small box centred at that point is taken out. Associated with C^l is a set of indices

$$\{j_l\} = \{a_1^l, \ldots, a_n^l, b_1^l, \ldots, b_n^l, c_1^{1,l}, \ldots, c_n^{1,l}, \ldots, c_1^{p,l}, \ldots, c_n^{p,l}, d_1^{1,l}, \ldots, d_n^{1,l}, \ldots, d_1^{p,l}, \ldots, d_n^{p,l}\}. \tag{3.4}$$

We will denote by $\{j_l\} \to \infty$,

$$a_1^l, \ldots, a_n^l, b_1^l, \ldots, b_n^l, c_1^{1,l}, \ldots, c_n^{1,l}, \ldots, c_1^{p,l}, \ldots, c_n^{p,l}, d_1^{1,l}, \ldots, d_n^{1,l}, \ldots, d_1^{p,l}, \ldots, d_n^{p,l} \to \infty \tag{3.5}$$

where all indices go to infinity independently of each other. Note that as $\{j_l\} \to \infty$, we recover $\mathbb{R}^n$ a.e. from C^l. We will denote by $\chi_{\{j_l\}}$ the characteristic function on C^l. Note that for $f \in L^2(\mathbb{R}^n)$,

$$\lim_{\{j_l\}\to\infty} \chi_{\{j_l\}} f = f \quad \text{a.e.} \tag{3.6}$$

where the limit in (3.6) is taken in the L^2 topology. Let us write

$$D_{\{J_l^h\}} = C^l \times \cdots \times C^h \qquad l \leqslant h. \tag{3.7}$$

Associated with $D_{\{J_l^h\}}$ is a set of indices

$$\{J_l^h\} = \bigcup_{\alpha=l}^{h} \{j_\alpha\} \tag{3.8}$$

and as before, we will use the notation $\{J_l^h\} \to \infty$ to mean

$$\{j_l\} \to \infty, \ldots, \{j_h\} \to \infty \tag{3.9}$$

where the indices are taken to infinity independently of each other. Finally, we will denote by $\int_{rO}$ Riemann or improper Riemann integration over the region O and by $\int_O$ Lebesgue integration over the region O.

Theorem 3.1. *Theorem 2.3 is true.*

Proof. Trotter's product formula (see [7, 8, 10]) and Schwarz's inequality imply that

$$\begin{aligned} &\int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y}, \eta) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) F_{\vec{x}_0}(\vec{z}, \gamma) \right](\vec{y})\, \mathrm{d}\vec{y} \\ &\quad = \lim_{k\to\infty} \int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y}, \eta) \left[\left\{ \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \right\}^k F_{\vec{x}_0}(\vec{z}, \gamma) \right](\vec{y})\, \mathrm{d}\vec{y} \end{aligned} \tag{3.10}$$

where the limit in (3.10) is taken pointwise as a function of t. To the right of each of the operators $\exp(-\mathrm{i}tH_0/k\hbar)$ in (3.10), we put in the identity operator $\lim_{\{j_l\}\to\infty} \chi_{\{j_l\}}$ for $1 \leqslant l \leqslant k$ in increasing order from right to left and the limit is taken in the L^2 topology. Since $\exp(-\mathrm{i}tV/k\hbar)$, $\exp(-\mathrm{i}tH_0/k\hbar)$, and multiplication by a characteristic function are all continuous operators, we can take all the limits outside of the operators and obtain

$$\begin{aligned} \left\{ \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \right\}^k F_{\vec{x}_0} &= \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \lim_{\{j_k\}\to\infty} \chi_{\{j_k\}} \cdots \\ &\quad \times \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \lim_{\{j_1\}\to\infty} \chi_{\{j_1\}} F_{\vec{x}_0} \\ &= \lim_{\{J_1^k\}\to\infty} \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \chi_{\{j_k\}} \cdots \\ &\quad \times \exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right) \exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) \chi_{\{j_1\}} F_{\vec{x}_0} \\ &= \lim_{\{J_1^k\}\to\infty} w_{n,k} \int_{D_{\{J_1^k\}}} \exp\left[\frac{\mathrm{i}\epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1)\right] F_{\vec{x}_0}(\vec{x}_1, \gamma)\, \mathrm{d}\vec{x}_1 \ldots \mathrm{d}\vec{x}_k. \end{aligned} \tag{3.11}$$

In the last equality in (3.11), we used the integral representation of the free evolution operator; we emphasize that all limits in (3.11) are taken in the L^2 topology and are taken independently of each other. Equations (3.6), (3.10), (3.11) and Schwarz's inequality imply that

$$\begin{aligned} \int_{\mathbb{R}^n} G_{\vec{x}}(\vec{y}, \eta) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) F_{\vec{x}_0}(\vec{z}, \gamma) \right](\vec{y})\, \mathrm{d}\vec{y} &= \lim_{k\to\infty} w_{n,k} \int_{\mathbb{R}^n} \left[\lim_{\{j_{k+1}\}\to\infty} \chi_{\{j_{k+1}\}} G_{\vec{x}}(\vec{x}_{k+1}, \eta) \right. \\ &\quad \left. \times \lim_{\{J_1^k\}\to\infty} w_{n,k} \int_{D_{\{J_1^k\}}} \exp\left[\frac{\mathrm{i}\epsilon}{\hbar} S_k(\vec{x}_{k+1}, \ldots, \vec{x}_1)\right] F_{\vec{x}_0}(\vec{x}_1, \gamma)\, \mathrm{d}\vec{x}_1 \ldots \mathrm{d}\vec{x}_k \right] \mathrm{d}\vec{x}_{k+1} \end{aligned}$$

$$= \lim_{k\to\infty} w_{n,k} \lim_{\{J_1^{k+1}\}\to\infty} \int_{D_{\{J_1^{k+1}\}}} G_{\vec{x}}\,(\vec{x}_{k+1},\eta)$$

$$\times \exp\left[\frac{\mathrm{i}\epsilon}{\hbar} S_k\,(\vec{x}_{k+1},\ldots,\vec{x}_1)\right] F_{\vec{x}_0}\,(\vec{x}_1,\gamma)\,\mathrm{d}\vec{x}_1\ldots\mathrm{d}\vec{x}_{k+1}. \tag{3.12}$$

In (3.12), all limits inside the integrals are taken independently of each other in the L^2 topology and all limits taken outside of the integral are taken pointwise in t. By construction, the integrand

$$G_{\vec{x}}\,(\vec{x}_{k+1},\eta)\exp\left[\frac{\mathrm{i}\epsilon}{\hbar} S_k\,(\vec{x}_{k+1},\ldots,\vec{x}_1)\right] F_{\vec{x}_0}(\vec{x}_1,\gamma) \tag{3.13}$$

is bounded and continuous on $D_{\{J_1^{k+1}\}}$. Hence, the Lebesgue integral over $D_{\{J_1^{k+1}\}}$ in the last equality of (3.12) can be replaced by a Riemann integral over $D_{\{J_1^{k+1}\}}$. Since the $\{J_1^{k+1}\}$ limits in (3.12) are all taken independently of each other, we can interpret those limits and the integral as an improper Riemann integral. □

Remark 3.2. It is not necessary to use improper Riemann integrals or put the discontinuities and singularities restriction on the potential. We forget about the holes centred at each element of K as given in (3.2) and take $C^l = A^l$ as defined in (3.1). Proceeding in the same manner as the above proof, we arrive at (3.12). At this point, we do not replace the Lebesgue integral with a Riemann integral since the integrand is not necessarily bounded and continuous over the region of integration. We are then left with an improper Lebesgue integral in which the convergence of the integral is taken pointwise in t.

4. Proof of theorem 2.1

In this section, we prove theorem 2.1. Let us denote

$$\begin{aligned} T^k &= \left\{\exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right)\exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right)\right\}^k \\ \bar{T}^k &= \left\{\exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right)\exp\left(\frac{-\mathrm{i}tV}{k\hbar}\right)\right\}^k. \end{aligned} \tag{4.1}$$

Theorem 4.1. *With our previously defined notation, we have*

$$|K(\vec{x},\vec{x}_0,\eta,\gamma,t)| \leqslant C_{t,\eta,\gamma} \tag{4.2}$$

where $C_{t,\eta,\gamma}$ is a constant depending only on t, η and γ.

Proof. Since the evolution operator has norm 1, using Schwarz's inequality on the kernel in (2.1) gives

$$|K(\vec{x},\vec{x}_0,\eta,\gamma,t)| \leqslant \|G_{\vec{x}}(\vec{y},\eta)\|_2\,\|F_{\vec{x}_0}(\vec{y},\gamma)\|_2 \equiv C_{t,\eta,\gamma}. \tag{4.3}$$

□

We will now show that $K(\vec{x},\vec{x}_0,\eta,\gamma,t,)$ is continuous as a function of $(\vec{x},\vec{x}_0)\in\mathbb{R}^n$.

Lemma 4.2. *Let $f, g \in L^2(\mathbb{R}^n)$, then the following is true:*

$$\int_{\mathbb{R}^n} g(\vec{x})\left[\exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right)f\right](\vec{x})\,\mathrm{d}\vec{x} = \int_{\mathbb{R}^n}\left[\exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right)g\right](\vec{x})f(\vec{x})\,\mathrm{d}\vec{x}. \tag{4.4}$$

Proof. Let χ_α be the characteristic function of the cube centred at the origin with sides of length α, then

$$\left[\exp\left(\frac{-\mathrm{i}\epsilon H_0}{\hbar}\right) f\right](\vec{x}) = \lim_{\alpha\to\infty} w_{n,1} \int_{\mathbb{R}^n} \chi_\alpha(\vec{y}) \exp\left[\frac{\mathrm{i}m\epsilon}{2\hbar}\left(\frac{\vec{x}-\vec{y}}{\epsilon}\right)^2\right] f(\vec{y})\,\mathrm{d}\vec{y} \tag{4.5}$$

where the limit in (4.5) is taken in the L^2 norm. Using Schwarz's inequality on α and Lebesgue's dominating convergence theorem on β, we have

$$\begin{aligned}\int_{\mathbb{R}^n} g(\vec{x}) &\left[\exp\left(\frac{-\mathrm{i}tH_0}{k\hbar}\right) f\right](\vec{x})\,\mathrm{d}\vec{x} \\ &= \lim_{\beta,\alpha\to\infty} w_{n,1} \int_{\mathbb{R}^n} \chi_\beta(\vec{x}) g(\vec{x}) \left\{\int_{\mathbb{R}^n} \chi_\alpha(\vec{y}) \exp\left[\frac{\mathrm{i}m\epsilon}{2\hbar}\left(\frac{\vec{x}-\vec{y}}{\epsilon}\right)^2\right] f(\vec{y})\,\mathrm{d}\vec{y}\right\} \mathrm{d}\vec{x} \\ &= \lim_{\beta,\alpha\to\infty} w_{n,1} \int_{\mathbb{R}^n} \chi_\alpha(\vec{y}) f(\vec{y}) \left\{\int_{\mathbb{R}^n} \chi_\beta(\vec{x}) \exp\left[\frac{\mathrm{i}m\epsilon}{2\hbar}\left(\frac{\vec{x}-\vec{y}}{\epsilon}\right)^2\right] g(\vec{x})\,\mathrm{d}\vec{x}\right\} \mathrm{d}\vec{y}\end{aligned} \tag{4.6}$$

where the limits are taken pointwise in t. Using Schwarz's inequality on β and Lebesgue's dominating convergence theorem on α in the last expression in (4.6) gives (4.4). □

Lemma 4.3. *Let $f, g \in L^2(\mathbb{R}^n)$, then the following is true:*

$$\int_{\mathbb{R}^n} g(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) f\right](\vec{x})\,\mathrm{d}\vec{x} = \int_{\mathbb{R}^n} \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) g\right](\vec{x}) f(\vec{x})\,\mathrm{d}\vec{x}. \tag{4.7}$$

Proof. Intuitively, if we think of the evolution as an exponential of the Hamiltonian $\bar{H}$, we can expand the exponential in powers of $\bar{H}$ and put all powers of $\bar{H}$ from the function f onto the function g since $\bar{H}$ is self-adjoint. This is also true for lemma 4.2.

Schwarz's inequality and Trotter's formula implies

$$\int_{\mathbb{R}^n} g(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) f\right](\vec{x})\,\mathrm{d}\vec{x} = \lim_{k\to\infty} \int_{\mathbb{R}^n} g(\vec{x}) \left[T^k f\right](\vec{x})\,\mathrm{d}\vec{x} \tag{4.8}$$

where the limit is taken pointwise in t and T^k is given in (4.1). Using lemma 4.2, and $\bar{T}^k$ as defined in (4.1), we obtain

$$\begin{aligned}\lim_{k\to\infty} \int_{\mathbb{R}^n} g(\vec{x}) \left[T^k f\right](\vec{x})\,\mathrm{d}\vec{x} &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \left[\bar{T}^k g\right](\vec{x}) f(\vec{x})\,\mathrm{d}\vec{x} \\ &= \int_{\mathbb{R}^n} \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right) g\right](\vec{x}) f(\vec{x})\,\mathrm{d}\vec{x}.\end{aligned} \tag{4.9}$$

□

Theorem 4.4. *With our previously defined notation, the expression*

$$\left|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)\right| \tag{4.10}$$

goes to zero as $(\vec{x}, \vec{x}_0)$ goes to $(\vec{y}, \vec{y}_0)$.

Proof. We first show that $K(\vec{x}, \vec{x}_0, \eta, \gamma, t)$ is separately continuous in $\vec{x}$ and $\vec{x}_0$, then jointly continuous. Schwarz's inequality implies that

$$\begin{aligned}
\left|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{x}_0, \eta, \gamma, t)\right|^2 &\leqslant \left\|G_{\vec{x}}(\vec{z}, \eta) - G_{\vec{y}}(\vec{z}, \eta)\right\|_2^2 \left\|F_{\vec{x}_0}(\vec{z}, \gamma)\right\|_2^2 \\
&= \left\|F_{\vec{x}_0}(\vec{z}, \gamma)\right\|_2^2 \left(\frac{m}{2\pi\hbar\eta}\right)^n \int_{\mathbb{R}^n} \left\{ \exp\left[\frac{-m\eta}{2\hbar}\left(\frac{\vec{x}-\vec{z}}{\eta}\right)^2\right] \right. \\
&\quad \left. - \exp\left[\frac{-m\eta}{2\hbar}\left(\frac{\vec{y}-\vec{z}}{\eta}\right)^2\right]\right\}^2 \mathrm{d}\vec{z} \\
&= C\left(\frac{m}{2\pi\hbar\eta}\right)^n \left\{ \int_{\mathbb{R}^n} 2\left[\exp\left(\frac{-m\eta}{2\hbar}\frac{\vec{z}^2}{\eta^2}\right)\right]^2 \mathrm{d}\vec{z} \right. \\
&\quad \left. - \int_{\mathbb{R}^n} 2\exp\left(\frac{-m\eta}{2\hbar}\frac{\vec{z}^2}{\eta^2}\right)\exp\left(\frac{-m\eta}{2\hbar}\frac{\left(\vec{z}+\vec{y}-\vec{x}\right)^2}{\eta^2}\right)\mathrm{d}\vec{z}\right\} \\
&= C_{\gamma,t}\, g\left(\vec{y}, \vec{x}\right)
\end{aligned} \tag{4.11}$$

where $C_{\gamma,t}$ is a constant independent of $\vec{x}_0$ and $g(\vec{y}, \vec{x})$ is independent of $\vec{x}_0$. Using Lebesgue's dominating convergence theorem on $\vec{x} \to \vec{y}$ in (4.11), we obtain $C_{\gamma,t}\, g(\vec{y}, \vec{x}) \to 0$. Using lemma 4.3 in equation (2.1), we can put the evolution operator on $G_{\vec{x}}(\vec{y}, \eta)$. With the same reasoning as (4.11), we obtain

$$\left|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{x}, \vec{y}_0, \eta, \gamma, t)\right|^2 \leqslant D_{\eta,t}\, f(\vec{y}_0, \vec{x}_0) \to 0 \tag{4.12}$$

as $\vec{x}_0 \to \vec{y}_0$.

Finally,

$$\begin{aligned}
\left|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)\right| &\leqslant \left|K(\vec{x}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{x}_0, \eta, \gamma, t)\right| \\
&\quad + \left|K(\vec{y}, \vec{x}_0, \eta, \gamma, t) - K(\vec{y}, \vec{y}_0, \eta, \gamma, t)\right| \\
&\leqslant \sqrt{C_{\gamma,t}\, g(\vec{y}, \vec{x})} + \sqrt{D_{\eta,t}\, f(\vec{y}_0, \vec{x}_0)} \to 0
\end{aligned} \tag{4.13}$$

as $(\vec{x}, \vec{x}_0) \to (\vec{y}, \vec{y}_0)$. □

5. Proof of theorems 2.2 and 2.4

We will now prove theorems 2.2 and 2.4.

Proposition 5.1. *Let $f, g \in L^2$, then*

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\vec{z}) \left[\exp\left(\frac{-\eta H_0}{\hbar}\right) g\right](\vec{z})\, \mathrm{d}\vec{z} &= \int_{\mathbb{R}^n} g(\vec{z}) \left[\exp\left(\frac{-\eta H_0}{\hbar}\right) f\right](\vec{z})\, \mathrm{d}\vec{z} \\
&= \int_{\mathbb{R}^{2n}} g(\vec{x}) G(\vec{x}, \vec{y}, \eta) f(\vec{y})\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{y}
\end{aligned} \tag{5.1}$$

where $G(\vec{x}, \vec{y}, \eta)$ is the Gaussian kernel given in equation (2.1).

Proof. Since $|f|$ and $|g|$ are in L^2, we have

$$\begin{aligned}
\int_{R^n} \left(\int_{R^n} |g(\vec{x})||G(\vec{x}, \vec{y}, \eta)||f(\vec{y})|\, \mathrm{d}\vec{x}\right) \mathrm{d}\vec{y} &= \int_{R^n} |g(\vec{x})| \left(\int_{R^n} G(\vec{x}, \vec{y}, \eta)|f(\vec{y})|\, \mathrm{d}\vec{x}\right) \mathrm{d}\vec{y} \\
&= \int_{\mathbb{R}^n} |g(\vec{x})| \left[\exp\left(\frac{-\eta H_0}{\hbar}\right)|f|\right](\vec{x})\, \mathrm{d}\vec{x} < \infty.
\end{aligned} \tag{5.2}$$

Equation (5.1) then follows from Fubini's theorem. □

Theorem 5.2. *Theorem 2.2(b) is true.*

Proof. First recall that in theorem 2.2(b), the wavefunctions $\phi, \psi \in L^2 \cap L^1$. Using lemma 4.3, the kernel in (2.1) can be written as

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = \left[\exp\left(\frac{-\gamma H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)G_{\vec{x}}(\vec{z}, \eta)\right](\vec{x}_0). \tag{5.3}$$

We have that

$$\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \lim_{\eta,\gamma\to 0}\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right](\vec{x}) \tag{5.4}$$

where the limits in (5.4) are taken in L^2 topology. Using Schwarz's inequality, the following holds:

$$\begin{aligned}\int_{\mathbb{R}^n}&\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}\\ &= \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}\end{aligned} \tag{5.5}$$

where the limits in (5.5) are taken pointwise in t. Lemma 4.3 and proposition 5.1 imply that

$$\begin{aligned}\Big[\exp&\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\Big](\vec{x})\\ &= \int_{\mathbb{R}^n} G(\vec{x}, \vec{y}, \eta)\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right](\vec{y})\,\mathrm{d}\vec{y}\\ &= \int_{\mathbb{R}^n}\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)G_{\vec{x}}(\vec{z}, \eta)\right](\vec{y})\left[\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right](\vec{y})\,\mathrm{d}\vec{y}\\ &= \int_{\mathbb{R}^n}\left[\exp\left(\frac{-\gamma H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)G_{\vec{x}}(\vec{z}, \eta)\right](\vec{x}_0)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\\ &= \int_{\mathbb{R}^n} K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0.\end{aligned} \tag{5.6}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x} &= \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^n}\phi(\vec{x})\left(\int_{\mathbb{R}^n}K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\right)\mathrm{d}\vec{x}\\ &= \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^{2n}}\phi(\vec{x})K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}\end{aligned} \tag{5.7}$$

where the last equality in (5.7) is obtained from theorem 2.1 (theorems 4.1 and 4.2) and the fact that the wavefunctions ϕ and ψ are in L^1. □

Theorem 5.3. *Theorem 2.2(a) is true.*

Proof. Let $C^1 = A^1, C^2 = A^2$ be as described in equation (3.1) and remark 3.2. Let $\chi^1_{\{j_1\}}, \chi^2_{\{j_2\}}$ be the characteristic function on the region C^1 and C^2, respectively ($\{j_1\}$ and $\{j_2\}$ are as described in equation (3.4) for C^1 and C^2). Let $\phi, \psi \in L^2$, then

$$\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right] = \lim_{\{j_1\}\to\infty}\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\chi^1_{\{j_1\}}\psi\right] \tag{5.8}$$

$$\phi = \lim_{\{j_2\}\to\infty}\chi^2_{\{j_1\}}\phi$$

where all limits in (5.8) are taken in the L^2 topology. Using (5.4), (5.5) and (5.8), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}\\
&\quad= \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}\\
&\quad= \lim_{\eta,\gamma\to 0}\int_{\mathbb{R}^n}\lim_{\{j_2\}\to\infty}\chi^2_{\{j_1\}}\phi(\vec{x})\\
&\qquad\times\lim_{\{j_2\}\to\infty}\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\chi^2_{\{j_2\}}\psi\right](\vec{x})\,\mathrm{d}\vec{x}.
\end{aligned} \tag{5.9}$$

Finally, using Schwarz's inequality and theorem 5.2 on the last expression in (5.9) gives

$$\begin{aligned}
\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x} &= \lim_{\eta,\gamma\to 0}\lim_{\{j_1\},\{j_2\}\to\infty}\int_{\mathbb{R}^n}\chi^1_{\{j_1\}}\phi(\vec{x})\\
&\quad\times\left[\exp\left(\frac{-\eta H_0}{\hbar}\right)\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\exp\left(\frac{-\gamma H_0}{\hbar}\right)\chi^2_{\{j_2\}}\psi\right](\vec{x})\,\mathrm{d}\vec{x}\\
&= \lim_{\eta,\gamma\to 0}\lim_{\{j_1\},\{j_2\}\to\infty}\int_{\mathbb{C}^1\times C^2}\phi(\vec{x})K(\vec{x},\vec{x}_0,\eta,\gamma,t)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}\\
&\equiv \lim_{\eta,\gamma\to 0}\bar{\int}_{\mathbb{R}^{2n}}\phi(\vec{x})K(\vec{x},\vec{x}_0,\eta,\gamma,t)\psi(\vec{x}_0)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}
\end{aligned} \tag{5.10}$$

where all limits outside of integrals are taken pointwise in t and $\bar{\int}$ by definition is an improper Lebesgue integral with convergence at infinity taken pointwise in t. □

Remark 5.4. If we restrict $\phi, \psi \in L^2$ to have a finite number of singularities and discontinuities as in the proof of theorem 2.3 (theorem 3.1), we can obtain an improper Riemann integral as opposed to an improper Lebesgue integral for theorem 5.3.

Theorem 5.5. *Theorem 2.4(a) is true.*

Proof. Equation (5.6) is true for all $\psi \in L^2$. Taken $\lim_{\eta,\gamma\to\infty}$ on both sides of (5.6) in the L^2 topology gives theorem 2.4(a), equation (2.6*a*). □

Theorem 5.6. *Theorem 2.4(b) is true.*

Proof. Let $\phi, \psi \in L^2$, then Schwarz's inequality and theorem 2.4(a) (theorem 5.4) imply

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x}) = \int_{\mathbb{R}^n} \phi(\vec{x}) \left(\lim_{\eta,\gamma\to 0} \int_{\mathbb{R}^n} \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\right) \mathrm{d}\vec{x}$$

$$= \lim_{\eta,\gamma\to 0} \int_{\mathbb{R}^n} \phi(\vec{x}) \left(\int_{\mathbb{R}^n} \psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\right) \mathrm{d}\vec{x} \tag{5.11}$$

where limits inside the integral are taken in L^2 and limits taken outside the integrals are taken pointwise in t. □

6. Proof of theorem 2.5

In this section, we prove theorem 2.5 and derive some properties of the tempered distribution in this theorem. Since we will be working with tempered distributions, we will let ϕ and ψ be in the class of rapidly decreasing test functions which we will denote by $S(\mathbb{R}^n)$. If ϕ and ψ are elements of $L^2(\mathbb{R}^n)$, we can choose a sequence of test functions $\{\phi_l\}$ and $\{\psi_j\}$ such that $\phi_l \to \phi$ and $\psi_j \to \psi$ in L^2. Applying Schwarz's inequality and using the fact that the evolution operator has an operator norm equal to 1, we can write

$$\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\, \mathrm{d}\vec{x} = \lim_{j,l\to\infty} \int_{\mathbb{R}^n} \phi_l(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi_j\right](\vec{x})\, \mathrm{d}\vec{x}. \tag{6.1}$$

Thus, by taking limits in the sense of (6.1), we can always recover all of the L^2 wavefunctions in the theory.

Theorem 6.1. *Theorem 2.5 is true.*

Proof. Suppose $\{\phi_k(\vec{x})\} \subset S(\mathbb{R}^n)$ with $\phi_k(\vec{x}) \to \phi(\vec{x})$ in $S(\mathbb{R}^n)$, then

$$\left|\int_{\mathbb{R}^n} [\phi(\vec{x}) - \phi_k(\vec{x})] \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\, \mathrm{d}\vec{x}\right|$$

$$\leqq \|\phi - \phi_k\|_2 \times \left\|\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right\|_2 \to 0. \tag{6.2}$$

Suppose $\{\psi_k(\vec{x})\} \subset S(\mathbb{R}^n)$with $\psi_k(\vec{x}) \to \psi(\vec{x})$ in $S(\mathbb{R}^n)$, then

$$\left|\int_{\mathbb{R}^n} \phi(\vec{x}) \left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)(\psi - \psi_k)\right](\vec{x})\, \mathrm{d}\vec{x}\right|$$

$$\leqq \|\phi\|_2 \times \left\|\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)(\psi - \psi_k)\right\|_2 = \|\phi\|_2 \times \|(\psi - \psi_k)\|_2 \to 0. \tag{6.3}$$

Hence the theorem follows from Schwartz's kernel theorem. □

Remark 6.2. Note that theorems 2.2 and 2.5 imply that

$$\lim_{\eta,\gamma\to\infty} \int_{\mathbb{R}^{2n}} \phi(\vec{x})\psi(\vec{x}_0) K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\, \mathrm{d}\vec{x}_0\, \mathrm{d}\vec{x} = \int K_t(\vec{x}, \vec{x}_0)\, \phi(\vec{x})\psi(\vec{x}_0)\, \mathrm{d}\vec{x}\, \mathrm{d}\vec{x}_0 \tag{6.4}$$

At $t = 0$, the evolution operator becomes the identity operator. In distributions language, We have the following:

Theorem 6.3. *At $t = 0$, $K_t(\vec{x}, \vec{x}_0)$ satisfies $K_0(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0)$.*

Proof. Since $\left[\exp\left(-\mathrm{i}t\bar{H}/\hbar\right)\psi\right](\vec{x}) = \psi(\vec{x})$ when $t = 0$, we have

$$\int K_0(\vec{x},\vec{x}_0)\,\phi(\vec{x})\psi\left(\vec{x}_0\right)\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x} = \int_{\mathbb{R}^n}\phi(\vec{x})\psi(\vec{x})\,\mathrm{d}\vec{x}$$

$$= \iint \phi(\vec{x})\delta(\vec{x}-\vec{x}_0)\,\psi\left(\vec{x}_0\right)\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0. \qquad (6.5)$$

We extend (6.5) to all of $S(\mathbb{R}^n\times\mathbb{R}^n)$. Let $\eta(\vec{x},\vec{x}_0)\in S(\mathbb{R}^n\times\mathbb{R}^n)$. Choose a sequence of functions $\eta_k(\vec{x},\vec{x}_0)$ in $S(\mathbb{R}^n\times\mathbb{R}^n)$ such that $\eta_k\to\eta$ in $S(\mathbb{R}^n\times\mathbb{R}^n)$ and $\eta_k = \sum_{i=0}^{j_k}u_{i,k}(\vec{x})v_{i,k}\left(\vec{x}_0\right)$, where $u_{i,k}$ and $v_{i,k}\in D(\mathbb{R}^n)$, the C^∞ compactly supported test functions, then

$$\int K_0(\vec{x},\vec{x}_0)\,\eta(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0 = \lim_{k\to\infty}\int K_0(\vec{x},\vec{x}_0)\,\eta_k(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0$$

$$= \lim_{k\to\infty}\int_{\mathbb{R}^n}\sum_{i=0}^{j_k}u_{i,k}(\vec{x})v_{i,k}(\vec{x})\,\mathrm{d}\vec{x}$$

$$= \int_{\mathbb{R}^n}\eta(\vec{x},\vec{x})\,\mathrm{d}\vec{x}$$

$$= \int\delta(\vec{x}-\vec{x}_0)\,\eta(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0. \qquad (6.6)$$

□

It is well known that the free propagator satisfies $K_t^{\mathrm{free}}(\vec{x},\vec{x}_0) = K_t^{\mathrm{free}}(\vec{x}_0,\vec{x})$. We will show a similar property for the tempered distribution in theorem 6.1. Intuitively, it is reasonable to believe that the from lemma 4.5 we can conclude the following.

Theorem 6.4. $K_t(\vec{x},\vec{x}_0) = K_t(\vec{x}_0,\vec{x})$ *where equality is in the sense of distributions.*

Proof. $\forall\phi(\vec{x}),\ \psi\left(\vec{x}_0\right)\in S(\mathbb{R}^n)$, we have that

$$\int K_t(\vec{x},\vec{x}_0)\,\phi(\vec{x})\psi\left(\vec{x}_0\right)\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0 = \int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}$$

$$= \int_{\mathbb{R}^n}\psi\left(\vec{x}_0\right)\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\phi\right]\left(\vec{x}_0\right)\mathrm{d}\vec{x}_0 = \int_{\mathbb{R}^n}\psi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\phi\right](\vec{x})\,\mathrm{d}\vec{x}$$

$$= \int K_t(\vec{x},\vec{x}_0)\,\psi(\vec{x})\phi\left(\vec{x}_0\right)\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0 = \int K_t(\vec{x}_0,\vec{x})\,\psi\left(\vec{x}_0\right)\phi(\vec{x})\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0. \qquad (6.7)$$

We extend (6.7) to all of $S(\mathbb{R}^n\times\mathbb{R}^n)$. Let $\eta(\vec{x},\vec{x}_0)\in S(\mathbb{R}^n\times\mathbb{R}^n)$. Choose a sequence of functions $\eta_k(\vec{x},\vec{x}_0)$ in $S\left(\mathbb{R}^{2n}\right)$ such that $\eta_k\to\eta$ in $S\left(\mathbb{R}^{2n}\right)$ and $\eta_k = \sum_{i=0}^{j_k}u_{i,k}(\vec{x})v_{i,k}\left(\vec{x}_0\right)$ where $u_{i,k}$ and $v_{i,k}\in D(\mathbb{R}^n)$, the C^∞ compactly supported test functions. We then have

$$\int K_t(\vec{x},\vec{x}_0)\,\eta(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0 = \lim_{k\to\infty}\int K_t(\vec{x},\vec{x}_0)\,\eta_k(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0$$

$$= \lim_{k\to\infty}\int K_t(\vec{x}_0,\vec{x})\,\eta_k(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0 = \int K_t(\vec{x}_0,\vec{x})\,\eta(\vec{x},\vec{x}_0)\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{x}_0. \qquad (6.8)$$

□

7. Proof of theorems 2.6 and 2.7

Theorem 7.1. *Theorem 2.6 is true.*

Proof. The non-standard equivalent of equation (2.8) is: for all $\omega \in {}^*\mathbb{N} - \mathbb{N}$,

$$K(\vec{x}, \vec{x}_0, \eta, \gamma, t) = st\left({}^*K_\omega(\vec{x}, \vec{x}_0, \eta, \gamma, t)\right). \tag{7.1}$$

□

Theorem 7.2. *Theorem 2.7 is true.*

Proof. Let

$$\begin{aligned}\bar{H}(\eta, \gamma) &= \bar{\int}_{\mathbb{R}^{2n}} \phi(\vec{x})\psi\left(\vec{x}_0\right)K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}\\ H(\eta, \gamma) &= \int_{\mathbb{R}^{2n}} \phi(\vec{x})\psi\left(\vec{x}_0\right)K(\vec{x}, \vec{x}_0, \eta, \gamma, t)\,\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}\\ C &= \int_{\mathbb{R}^n} \phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}.\end{aligned} \tag{7.2}$$

Theorem 2.2 implies that for all $\epsilon \in \mathbb{R}^+$, there exists a $\delta \in \mathbb{R}^+$ such that

$$|H(\eta, \gamma) - C| < \epsilon \qquad \text{when} \quad \eta, \gamma < \delta. \tag{7.3}$$

We now *-transform (7.3) and conclude that any positive infinitesimal η and γ is less than δ. Since we can do this for any standard ϵ, theorem 2.7 holds for H. A similar argument shows that the theorem is also true for $\bar{H}$ □

8. The harmonic oscillator

We now compute the harmonic oscillator propagator for $0 < t < \pi/\lambda$ using the formulae above. Some of the techniques that we will use was worked out previously in [19], for full details, we will occasionally refer the reader to [19]. For the harmonic oscillator, equation (1.5) reads (with a shift in the indices)

$$\begin{aligned}K\left(\vec{q}, \vec{q}_0, \eta, \gamma, t\right) = \lim_{k\to\infty} w_{n,k+1} \int_{r\mathbb{R}^{(k+2)n}} F_{\vec{q}_0}\left(\vec{x}_0, \gamma\right) G_{\vec{q}}\left(\vec{x}_{k+1}, \eta\right)\\ \times \exp\left\{\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{k+1}\left[\frac{m}{2}\left(\frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon}\right)^2 - \frac{m}{2}\lambda^2\left(\vec{x}_j\right)^2\right]\right\}\mathrm{d}\vec{x}_0\ldots\mathrm{d}\vec{x}_{k+1}.\end{aligned} \tag{8.1}$$

Let us write $\vec{x}_j = \left(x_j^1, \ldots, x_j^n\right)$, and

$$\begin{aligned}&\exp\left\{\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{k+1}\left[\frac{m}{2}\left(\frac{\vec{x}_j - \vec{x}_{j-1}}{\epsilon}\right)^2 - \frac{m}{2}\lambda^2\left(\vec{x}_j\right)^2\right]\right\}\\ &\qquad = \prod_{\alpha=1}^{n}\exp\left\{\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{k+1}\left[\frac{m}{2}\left(\frac{x_j^\alpha - x_{j-1}^\alpha}{\epsilon}\right)^2 - \frac{m}{2}\lambda^2\left(x_j^\alpha\right)^2\right]\right\}.\end{aligned} \tag{8.2}$$

The popular method to compute the time-sliced harmonic oscillator path integral is to use (8.2) to decouple the integrals in 1.0 and reduce the problem to one of producing one-dimensional harmonic oscillators. Due to the extra $\vec{x}_0, \vec{x}_{k+1}$ integrals in (8.1), it is not immediately clear that we can use (8.2) to decouple the improper Riemann integrals.

For the moment, let us consider just one of the entries in the product of (8.2). To shorten the notation, let us write

$$
\begin{aligned}
\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{k+1}\left[\frac{m}{2}\left(\frac{x_j^\alpha - x_{j-1}^\alpha}{\epsilon}\right)^2 - \frac{m}{2}\lambda^2 (x_j^\alpha)^2\right] &= \left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\Bigg[\left(x_0^\alpha\right)^2 - 2x_0^\alpha x_1^\alpha + \left(x_{k+1}^\alpha\right)^2 - 2x_k^\alpha x_{k+1}^\alpha \\
&\quad + \sum_{j=1}^{k} 2\left(x_j^\alpha\right)^2 - \sum_{j=1}^{k} 2x_j^\alpha x_{j-1}^\alpha - \epsilon^2\lambda^2 \sum_{j=1}^{k+1}\left(x_j^\alpha\right)^2\Bigg] \\
&= \left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{x}^\alpha)^t \left\{ \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & & & & \vdots \\ 0 & & 0 & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & & -1 & 1 \end{pmatrix} \right. \\
&\quad - \epsilon^2\lambda^2 \begin{pmatrix} 0 & \cdots & & & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \;\; 1 & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 \end{pmatrix} \\
&\quad \left. + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \right\} \vec{x}^\alpha \\
&= \left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left((\vec{x}^\alpha)^t\, T_k \vec{x}^\alpha\right) \qquad (8.3)
\end{aligned}
$$

where T_k is the $(k+2) \times (k+2)$ symmetric matrix,

$$T_k = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & & & & \vdots \\ 0 & & S_k & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & & -1 & 1-\epsilon^2\lambda^2 \end{pmatrix} \tag{8.4}$$

with S_k being the $k \times k$ symmetric matrix $S_k = A_k - \epsilon^2\lambda^2 B_k$, where

$$A_k = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$
$$B_k = \begin{pmatrix} 1 & 0 & \cdots & & & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 \\ 0 & \cdots & & & & \cdots & 0 & 1 \end{pmatrix} \tag{8.5}$$

and $\vec{x}^\alpha$ is the column vector

$$\vec{x}^\alpha = \begin{pmatrix} x_0^\alpha \\ x_1^\alpha \\ \vdots \\ x_k^\alpha \\ x_{k+1}^\alpha \end{pmatrix}. \tag{8.6}$$

Let $w^\alpha : [0,t] \to \mathbb{R}$ be such that $|w^\alpha(s)| < \infty$ and $w^\alpha(0) = x_0^\alpha$, $w^\alpha(t) = x_{k+1}^\alpha$. In the literature, the path w^α is usually taken to be the path of the classical harmonic oscillator. Here, we allow w^α to be any finite path that starts at x_0^α and ends at x_{k+1}^α. We do not assume prior knowledge of classical mechanics. We make the substitution $x_j^\alpha = w^\alpha\,(jt/(k+1)) + y_j^\alpha = w_j^\alpha + y_j^\alpha$ (note that $y_0^\alpha = 0 = y_{k+1}^\alpha$ since $w^\alpha(0) = x_0^\alpha$ and $w^\alpha(t) = x_{k+1}^\alpha$). Using the fact that

T_k is symmetric, we have

$$
\begin{aligned}
(\vec{x}^\alpha)^t\, T_k \vec{x}^\alpha &= (\vec{y}^\alpha + \vec{w}^\alpha)^t\, T_k\, (\vec{y}^\alpha + \vec{w}^\alpha) \\
&= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{y}^\alpha + (\vec{w}^\alpha)^t\, T_k \vec{y}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{w}^\alpha \\
&= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{y}^\alpha + (T_k \vec{w}^\alpha)^t\, \vec{y}^\alpha + \left((\vec{w}^\alpha)^t\, T_k \vec{y}^\alpha\right)^t \\
&= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{y}^\alpha + (T_k \vec{w}^\alpha)^t\, \vec{y}^\alpha + (\vec{w}^\alpha)^t\, T_k \vec{y}^\alpha \\
&= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{y}^\alpha + 2\,(T_k \vec{w}^\alpha)^t\, \vec{y}^\alpha
\end{aligned}
\tag{8.7}
$$

where

$$
\vec{y}^\alpha = \begin{pmatrix} 0 \\ y_1^\alpha \\ \vdots \\ y_k^\alpha \\ 0 \end{pmatrix} \qquad \vec{w}^\alpha = \begin{pmatrix} w_0^\alpha = x_0^\alpha \\ w_1^\alpha \\ \vdots \\ w_k^\alpha \\ w_{k+1}^\alpha = x_{k+1}^\alpha \end{pmatrix}. \tag{8.8}
$$

By using $y_0^\alpha = 0 = y_{k+1}^\alpha$ and writing T_k as

$$
T_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ 0 & & S_k & & 0 \\ \vdots & & & & 0 \\ 0 & \cdots & & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & & & & \vdots \\ 0 & & 0 & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & & -1 & 1-\epsilon^2\lambda^2 \end{pmatrix} \tag{8.9}
$$

we obtain

$$
\begin{aligned}
(\vec{x}^\alpha)^t\, T_k \vec{x}^\alpha &= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + (\vec{y}^\alpha)^t\, T_k \vec{y}^\alpha + 2\,(T_k \vec{w}^\alpha)^t\, \vec{y}^\alpha \\
&= (\vec{w}^\alpha)^t\, T_k \vec{w}^\alpha + \left(\hat{y}^\alpha\right)^t\, S_k \hat{y}^\alpha + 2\,(\vec{\rho}^\alpha)^t\, \hat{y}^\alpha
\end{aligned}
\tag{8.10}
$$

where

$$
\hat{y}^\alpha = \begin{pmatrix} y_1^\alpha \\ \vdots \\ y_k^\alpha \end{pmatrix} \qquad \vec{\rho}^\alpha = S_k \begin{pmatrix} w_1^\alpha \\ \vdots \\ w_{k-1}^\alpha \\ w_k^\alpha \end{pmatrix} - \begin{pmatrix} w_0^\alpha = x_0^\alpha \\ 0 \\ \vdots \\ 0 \\ w_{k+1}^\alpha = x_{k+1}^\alpha \end{pmatrix} = S_k \hat{w}^\alpha - \hat{\hat{w}}^\alpha. \tag{8.11}
$$

Lemma 8.1. *Let $t \in \mathbb{R}$ and $0 < t < \pi/\lambda$. For any $\omega \in {}^*\mathbb{N} - \mathbb{N}$, ${}^*S_\omega$ is positive definite in the $*$-transformed sense. Here, ${}^*S_\omega$ is the $*$-transform of the matrix S_k defined after equation (8.4).*

Proof. See [19]. □

We now go back to equations (8.1) and (8.2) in non-standard analysis form. With an abuse of notation, in non-standard analysis (8.1) reads

$$
\begin{aligned}
K(\vec{q},\vec{q}_0,\eta,\gamma,t) &= st\Bigg\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\\
&\quad\times\exp\left\{\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{\omega+1}\left[\frac{m}{2}\left(\frac{\vec{x}_j-\vec{x}_{j-1}}{\epsilon}\right)^2-\frac{m}{2}\lambda^2\left(\vec{x}_j\right)^2\right]\right\}\mathrm{d}\vec{x}_0\ldots\mathrm{d}\vec{x}_{\omega+1}\Bigg\}\\
&= st\Bigg\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\frac{\mathrm{i}\epsilon}{\hbar}\sum_{j=1}^{\omega+1}\left[\frac{m}{2}\left(\frac{x_j^\alpha-x_{j-1}^\alpha}{\epsilon}\right)^2-\frac{m}{2}\lambda^2\left(x_j^\alpha\right)^2\right]\right\}\mathrm{d}\vec{x}_0\ldots\mathrm{d}\vec{x}_{\omega+1}\Bigg\}\\
&= st\Bigg\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left((\vec{x}^\alpha)^t\,T_\omega\vec{x}^\alpha\right)\right\}\mathrm{d}\vec{x}_0\ldots\mathrm{d}\vec{x}_{\omega+1}\Bigg\}.
\end{aligned}
\tag{8.12}
$$

We perform a *-transform of the change of variables described in equation (8.7) on $\vec{x}_1\ldots\vec{x}_\omega$, and obtain

$$
\begin{aligned}
K(\vec{q},\vec{q}_0,\eta,\gamma,t) &= st\Bigg\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{w}^\alpha)^t\,T_\omega\vec{w}^\alpha\right\}\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[\left(\hat{y}^\alpha\right)^t S_\omega\hat{y}^\alpha+2\,(\vec{\rho}^\alpha)^t\,\hat{y}^\alpha\right]\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}_{\omega+1}\,\mathrm{d}\vec{y}_1\ldots\mathrm{d}\vec{y}_\omega\Bigg\}.
\end{aligned}
\tag{8.13}
$$

Since S_ω is positive definite, it is invertible. Since S_ω is symmetric, the following is true:

$$
\left(\hat{y}^\alpha\right)^t S_\omega\hat{y}^\alpha+2\,(\vec{\rho}^\alpha)^t\,\hat{y}^\alpha=\left(\hat{y}^\alpha+S_\omega^{-1}\vec{\rho}^\alpha\right)^t S_\omega\left(\hat{y}^\alpha+S_\omega^{-1}\vec{\rho}^\alpha\right)-(\vec{\rho}^\alpha)^t\,S_\omega^{-1}\vec{\rho}^\alpha.
\tag{8.14}
$$

Using (8.14) in (8.13) and performing the transformation $z_j^\alpha=y_j^\alpha+\left(S_\omega^{-1}\vec{\rho}^\alpha\right)_j$, we obtain

$$
\begin{aligned}
K(\vec{q},\vec{q}_0,\eta,\gamma,t) &= st\Bigg\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_\omega\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_\omega^{-1}\vec{\rho}^\alpha\right]\right\}\\
&\quad\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{z}^\alpha)^t\,S_\omega\vec{z}^\alpha\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}_{\omega+1}\,\mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_\omega\Bigg\}.
\end{aligned}
\tag{8.15}
$$

Note that before the improper limits are taken on the integrals, the limits of integration on the variables z_j^α are dependent on $\vec{x}_0$ and $\vec{x}_{\omega+1}$ due to the fact that $\vec{\rho}^\alpha$ is dependent on them. Thus, we still cannot decouple the improper Riemann integrals.

Let us take a look at the limits of integration more closely. Before the improper limits on the integrals are taken in equation (8.15), we have

$$w_{n,\omega+1}\int_{\mathcal{O}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_\omega\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_\omega^{-1}\vec{\rho}^\alpha\right]\right\}$$
$$\times\left\{\int_{\bar{\mathcal{O}}}\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{z}^\alpha)^t\,S_\omega\vec{z}^\alpha\right\}\mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_\omega\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}_{\omega+1}\qquad(8.16)$$

where both $\mathcal{O}$ and $\bar{\mathcal{O}}$ are *-compact and the boundary of $\bar{\mathcal{O}}$ depends on $\vec{x}_0$, $\vec{x}_{\omega+1}$ and a set of indices $\{J\}$ such that as $\{J\}\to\infty$, $\bar{\mathcal{O}}\to{}^*\mathbb{R}^{\omega n}$ in the *-transformed sense. The reason for the $\bar{\mathcal{O}}$ dependence on $\vec{x}_0$, $\vec{x}_{\omega+1}$ is due to the fact that $\vec{\rho}^\alpha$ is dependent on them (equation (8.11)) and we performed the change of variables from equation (8.13) to equation (8.15). Furthermore, the boundary of $\mathcal{O}$ is also indexed by a set similar to that of $\{J\}$. What we would like to do is pass the $\{J\}$ limits inside the $\mathcal{O}$ integral and decouple the improper Riemann integrals in equation (8.15) into improper Riemann integrals in $\mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_\omega$ then $\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}_{\omega+1}$.

It is well known from the one-dimensional harmonic oscillator that

$$w_{1,k+1}\int_{r\mathbb{R}^k}\exp\left(\frac{\mathrm{i}m}{2\hbar\epsilon}(z^\alpha)^t\,S_kz^\alpha\right)\mathrm{d}z_1^\alpha\ldots\mathrm{d}z_k^\alpha=\left(\frac{m}{2\pi\mathrm{i}\hbar\epsilon}\right)^{1/2}\sqrt{\frac{1}{\det S_k}}.\qquad(8.17)$$

Let us fix an $\mathcal{O}$. Since $\mathcal{O}$ is compact (see the construction in equation (3.12)). We can *-transform and conclude from (8.16) and (8.17) that for any $\beta\in{}^*\mathbb{R}^+$, there exists a fixed $M\in{}^*\mathbb{R}^+$ that depends only on $\mathcal{O}$ such that

$$\left|w_{n,\omega+1}\int_{\bar{\mathcal{O}}}\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{z}^\alpha)^t\,S_\omega\vec{z}^\alpha\right\}\mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_\omega-\left(\left(\frac{m}{2\pi\mathrm{i}\hbar\epsilon}\right)^{1/2}\sqrt{\frac{1}{\det S_\omega}}\right)^n\right|<\beta\qquad(8.18)$$

whenever all entries of $\{J\}$ are bigger than M. In other words, because $\mathcal{O}$ is compact, for all $(\vec{x}_0,\vec{x}_{\omega+1})\in\mathcal{O}$, equation (8.18) is true whenever all entries of $\{J\}$ are bigger than a fixed M; furthermore, this M depends on $\mathcal{O}$.

Equation (8.18) allows us to use the *-Lebesgue dominating convergence theorem and pass the $\{J\}$ limits inside the $\mathcal{O}$ integral and decouples the improper Riemann integrals. Thus, we have proved the following:

Theorem 8.2. *For the harmonic oscillator,*

$$K(\vec{q},\vec{q}_0,\eta,\gamma,t)=st\left\{w_{n,\omega+1}\int_{r\mathbb{R}^{(\omega+2)n}}F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\right.$$
$$\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_\omega\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_\omega^{-1}\vec{\rho}^\alpha\right]\right\}$$
$$\left.\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{z}^\alpha)^t\,S_\omega\vec{z}^\alpha\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{x}_{\omega+1}\,\mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_\omega\right\}$$

$$
\begin{aligned}
&= st\Bigg\{ w_{n,\omega+1}\Bigg\{ \int_{r\mathbb{R}^{2n}} F_{\vec{q}_0}\,(\vec{x}_0,\gamma)\, G_{\vec{q}}\,(\vec{x}_{\omega+1},\eta) \\
&\quad \times \prod_{\alpha=1}^{n} \exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^{\alpha})^{t}\, T_{\omega}\vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^{t}\, S_{\omega}^{-1}\vec{\rho}^{\alpha}\right]\right\} \mathrm{d}\vec{x}_0\,\vec{x}_{\omega+1}\Bigg\} \\
&\quad \times \int_{r\mathbb{R}^{\omega n}} \prod_{\alpha=1}^{n} \exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{z}^{\alpha})^{t}\, S_{\omega}\vec{z}^{\alpha}\right\} \mathrm{d}\vec{z}_1\ldots\mathrm{d}\vec{z}_{\omega}\Bigg\} \\
&= st\Bigg\{ \left(\left(\frac{m}{2\pi\mathrm{i}\hbar\epsilon}\right)^{1/2}\sqrt{\frac{1}{\det S_{\omega}}}\right)^{n} \int_{r\mathbb{R}^{2n}} F_{\vec{q}_0}\,(\vec{x}_0,\gamma)\, G_{\vec{q}}\,(\vec{x}_{\omega+1},\eta) \\
&\quad \times \prod_{\alpha=1}^{n} \exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^{\alpha})^{t}\, T_{\omega}\vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^{t}\, S_{\omega}^{-1}\vec{\rho}^{\alpha}\right]\right\} \mathrm{d}\vec{x}_0\,\vec{x}_{\omega+1}\Bigg\}.
\end{aligned}
\tag{8.19}
$$

Proof. See above. □

It now remains to compute the last equality in (8.19).

Proposition 8.3. *With the previously defined notation,*

$$
st\left\{\left(\left(\frac{m}{2\pi\mathrm{i}\hbar\epsilon}\right)^{1/2}\sqrt{\frac{1}{\det S_{\omega}}}\right)^{n}\right\} = \left(\frac{m}{2\pi\mathrm{i}\hbar}\right)^{n/2}\left(\frac{\lambda}{\sin\lambda t}\right)^{n/2}. \tag{8.20}
$$

Proof. See [19]. □

Proposition 8.4. *Let $\vec{x}_0, \vec{x}_n = \vec{y} \in \mathbb{R}^n$ be fixed. With the previously defined notation,*

$$
\begin{aligned}
&\lim_{k\to\infty}\left\{\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^{\alpha})^{t}\, T_{k}\vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^{t}\, S_{k}^{-1}\vec{\rho}^{\alpha}\right]\right\}\right\} \\
&\qquad = \exp\left\{\frac{\mathrm{i}m}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^{2}+\vec{y}^{2}\right)\cos\lambda t - 2\vec{y}\vec{x}_0\right]\right\}.
\end{aligned}
\tag{8.21}
$$

Remark 8.5. Note that proposition 8.4 does not take place in the non-standard world.

Proof. This is just the classical version of the non-standard results obtained from [19] (see [19] for more details). □

Proposition 8.6. *With the previously defined notation,*

$$
\begin{aligned}
&st\left\{\int_{r\mathbb{R}^{2n}} F_{\vec{q}_0}\,(\vec{x}_0,\gamma)\, G_{\vec{q}}\,(\vec{x}_{\omega+1},\eta)\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)(\vec{w}^{\alpha})^{t}\, T_{\omega}\vec{w}^{\alpha} - (\vec{\rho}^{\alpha})^{t}\, S_{\omega}^{-1}\vec{\rho}^{\alpha}\right\}\mathrm{d}\vec{x}_0\,\vec{x}_{\omega+1}\right\} \\
&\qquad = \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}\,(\vec{x}_0,\gamma)\, G_{\vec{q}}(\vec{y},\eta)\exp\left\{\frac{\mathrm{i}m}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^{2}+\vec{y}^{2}\right)\cos\lambda t - 2\vec{y}\vec{x}_0\right]\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{y}.
\end{aligned}
\tag{8.22}
$$

Proof. Using proposition 8.4 and Lebesgue's dominating convergence theorem, we obtain

$$st\left\{\int_{r\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{x}_{\omega+1},\eta)\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_\omega\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_\omega^{-1}\vec{\rho}^\alpha\right]\right\}\mathrm{d}\vec{x}_0\,\vec{x}_{\omega+1}\right\}$$

$$= \lim_{k\to\infty}\int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{y},\eta)$$

$$\times\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_k\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_k^{-1}\vec{\rho}^\alpha\right]\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{y}$$

$$= \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{y},\eta)$$

$$\times\lim_{k\to\infty}\left\{\prod_{\alpha=1}^{n}\exp\left\{\left(\frac{\mathrm{i}m}{2\hbar\epsilon}\right)\left[(\vec{w}^\alpha)^t\,T_k\vec{w}^\alpha-(\vec{\rho}^\alpha)^t\,S_k^{-1}\vec{\rho}^\alpha\right]\right\}\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{y}$$

$$= \int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{y},\eta)\exp\left\{\frac{\mathrm{i}m}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^2+\vec{y}^2\right)\cos\lambda t-2\vec{y}\vec{x}_0\right]\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{y}. \tag{8.23}$$

□

Theorem 8.7. *For the harmonic oscillator,*

$$K(\vec{q},\vec{q}_0,\eta,\gamma,t)=\left(\frac{m}{2\pi\mathrm{i}\hbar}\right)^{n/2}\left(\frac{\lambda}{\sin\lambda t}\right)^{n/2}\int_{\mathbb{R}^{2n}} F_{\vec{q}_0}(\vec{x}_0,\gamma)\,G_{\vec{q}}(\vec{y},\eta)$$

$$\times\exp\left\{\frac{\mathrm{i}m}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{x}_0^2+\vec{y}^2\right)\cos\lambda t-2\vec{y}\vec{x}_0\right]\right\}\mathrm{d}\vec{x}_0\,\mathrm{d}\vec{y}. \tag{8.24}$$

Proof. Equation (8.24) follows from theorem 8.2, proposition 8.3 and proposition 8.5. □

Theorem 8.8. *Let $\phi,\psi\in L^1\cap L^2$, then for the harmonic oscillator Hamiltonian and for $0<t<\pi/\lambda$,*

$$\int_{\mathbb{R}^n}\phi(\vec{x})\left[\exp\left(\frac{-\mathrm{i}t\bar{H}}{\hbar}\right)\psi\right](\vec{x})\,\mathrm{d}\vec{x}=\left(\frac{m}{2\pi\mathrm{i}\hbar}\right)^{n/2}\left(\frac{\lambda}{\sin\lambda t}\right)^{n/2}$$

$$\times\int_{\mathbb{R}^{2n}}\phi(\vec{q}_0)\psi(\vec{q})\exp\left\{\frac{\mathrm{i}m}{\hbar}\frac{\lambda}{\sin\lambda t}\left[\left(\vec{q}_0^2+\vec{q}^2\right)\cos\lambda t-2\vec{q}\vec{q}_0\right]\right\}\mathrm{d}\vec{q}_0\,\mathrm{d}\vec{q}. \tag{8.25}$$

Proof. Note that $|K(\vec{q},\vec{q}_0,\eta,\gamma,t)|\leqslant C_{n,\lambda,t}$. Substituting equation (8.24) in equation (1.3) and using the Lebesgue dominating theorem on the η,γ limits give (8.25). □

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